JACOB'S LADDERS AND THE ASYMPTOTIC FORMULA FOR SHORT AND MICROSCOPIC PARTS OF THE HARDY-LITTLEWOOD INTEGRAL OF THE FUNCTION $|\zeta(1/2+it)|^4$

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Abstract. The elementary geometric properties of Jacob's ladders of the second order lead to a class of new asymptotic formulae for short and microscopic parts of the Hardy-Littlewood integral of $|\zeta(1/2+it)|^4$. These formulae cannot be obtained by methods of Balasubramanian, Heath-Brown and Ivic.

1. Formulation of the Theorem

1.1. Let us remind that Hardy and Littlewood started to study the following integral in 1922

$$\int_1^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 \mathrm{d}t = \int_1^T Z^4(t)\mathrm{d}t, \tag{1.1}$$

where

$$Z(t)=e^{i\vartheta(t)}\zeta\left(\frac{1}{2}+it\right),\ \vartheta(t)=-\frac{1}{2}t\ln\pi+\mathrm{Im}\ln\Gamma\left(\frac{1}{4}+i\frac{t}{2}\right),$$

and they derived the following estimate (see [2], pp. 41, 59, [14], p. 124)

$$\int_1^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 \mathrm{d}t = \mathcal{O}(T\ln^4 T). \tag{1.2}$$

Let us remind furthermore that Ingham, in 1926, has derived the first asymptotic formula

$$\int_1^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 \mathrm{d}t = \frac{1}{2\pi^2}T\ln^4 T+\mathcal{O}(T\ln^3 T) \tag{1.3}$$

(see [3], p. 277, [14], p.125). In 1928 Titchmarsh has discovered a new treatment to the integral (1.1)

$$\int_0^T Z^4(t)e^{-\delta t}\mathrm{d}t \sim \frac{1}{2\pi^2}\frac{1}{\delta}\ln^4\frac{1}{\delta} \Rightarrow \int_1^T Z^4(t)\mathrm{d}t \sim \frac{1}{2\pi^2}T\ln^4 T \tag{1.4}$$

(see [14], pp. 136, 143). Let us remind, finally, the Titchmarsh-Atkinson formula (see [14], p. 145)

$$\int_0^T Z^4(t)e^{-\delta t}\mathrm{d}t = \frac{1}{\delta}\left(A\ln^4\frac{1}{\delta}+B\ln^3\frac{1}{\delta}+C\ln^2\frac{1}{\delta}+\right. \\ \left.+D\ln\frac{1}{\delta}+E\right)+\mathcal{O}\left\{\left(\frac{1}{\delta}\right)^{13/14+\epsilon}\right\},\ A=\frac{1}{2\pi^2}, \tag{1.5}$$

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which improved the Titchmarsh formula (1.4), and the Ingham - Heath-Brown formula (see [4], p. 129)

$$\int_0^T Z^4(t)\mathrm{d}t = T\sum_{K=0}^{4} C_K \ln^{4-K} T + \mathcal{O}(T^{7/8+\epsilon}),\ C_0 = \frac{1}{2\pi^2} \tag{1.6}$$

which improved the Ingham formula (1.3).

1.2. It is clear that the asymptotic formulae for short and microscopic parts

$$\int_T^{T+U} \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 \mathrm{d}t \tag{1.7}$$

of the Hardy-Littlewood integral (1.1) cannot be obtained by methods which lead to the results (1.2)-(1.6). It is proved in this paper that the Jacob's ladders of the second order $\varphi_2(T)$ (see [12]) lead to new asymptotic formulae in this direction.

Let us remind our formula (see [12], (5.11))

$$\begin{gathered} Z^4(t) = \frac{1}{2\pi^2}\left\{1+\mathcal{O}\left(\frac{(\ln\ln T)^2}{\ln T}\right)\right\}\ln^4 T\frac{\mathrm{d}\varphi_2(t)}{\mathrm{d}t}, \\ t\in[T,T+U_0],\ U_0 = T^{13/4+2\epsilon}. \end{gathered} \tag{1.8}$$

Then from (1.8) the multiplicative asymptotic formula for short and microscopic parts (1.7) of the Hardy-Littlewood integral (1.1) follows (compare [7], (1.2)).

Theorem.

$$\begin{gathered} \int_T^{T+U} Z^4(t)\mathrm{d}t = \frac{1}{2\pi^2}\left\{1+\mathcal{O}\left(\frac{(\ln\ln T)^2}{\ln T}\right)\right\} U\ln^4 T\tan[\alpha_2(T,U)], \\ U\in(0,U_0],\ U_0 = T^{13/14+2\epsilon}, \end{gathered} \tag{1.9}$$

where α_2 is the angle of the chord of the curve $y=\varphi_2(T)$ that binds the points $[T,\varphi_2(T)],\ [T+U,\varphi_2(T+U)]$.

Remark 1. The small improvements of the exponent $13/14$ of the type $13/14\to 8/9\to\dots$ are irrelevant in this question.

This paper is a continuation of the series of papers [5]-[13].

2. Some canonical equivalences

2.1. Let us remind that

$$\tan[\alpha_2(T,U_0)] = 1+\mathcal{O}\left(\frac{1}{\ln T}\right) \tag{2.1}$$

is true (see [12], (5.6)). Then, similarly to [7], 2.1, we call the chord binding the points $[T,\varphi_2(T)],\ [T+U_0,\varphi_2(T+U_0)]$ of the Jacob's ladder $y=\varphi_2(T)$ the *fundamental chord* (compare [7]).

Let us consider the set of all segments $[M,N]\subset[T,T+U_0]$.

Definition. The chord binding the points

$$[N,\varphi_2(N)],[M,\varphi_2(M)],\ [M,N]\subset[T,T+U_0],$$

such that the property

$$\tan[\alpha_2(N,M-N)]=1+o(1),\ T\to\infty \tag{2.2}$$

is fulfilled, is called the *almost parallel chord* to the fundamental chord. This property will be denoted by the symbol $/\!\!/_2$, (comp. [7]).

Now, we obtain the following corollary from (1.9) and (2.2).

Corollary 1. Let $[M,N]\subset[T,T+U_0]$. Then

$$\frac{1}{M-N}\int_N^M Z^4(t)\mathrm{d}t\sim\frac{1}{2\pi^2}\ln^4 T \ \Leftrightarrow\ /\!\!/_2. \tag{2.3}$$

Remark 2. We see that the analytic property

$$\frac{1}{M-N}\int_N^M Z^4(t)\mathrm{d}t\sim\frac{1}{2\pi^2}\ln^4 T$$

is equivalent to the geometric property $/\!\!/_2$ of Jacob's ladder $y=\varphi_2(T)$ of the second order.

2.2. Next, similarly to the case of the paper [7], the following corollary is obtained from our Theorem.

Corollary 2. There is a continuum of intervals $[M,N]\subset[T,T+U_0]$ such that the asymptotic formula

$$\int_N^M Z^4(t)\mathrm{d}t\sim\frac{1}{2\pi^2}(M-N)\ln^4 T \tag{2.4}$$

holds true.

Remark 3. Especially, there is a continuum of intervals $[N,M]:\ 0<M-N<1$, such that the asymptotic formula (2.4) is true (this follows from the elementary mean-value theorem of differentiation).

3. On microscopic parts of the Hardy-Littlewood integral (1.1) in neighborhoods of zeroes of the function $\zeta(1/2+iT)$

Let γ,γ' be a pair of neighboring zeroes of the function $\zeta(1/2+iT)$. The function $\varphi_2(T)$ is necessarily convex on some right neighborhood of the point $T=\gamma$, and this function is necessarily concave on some left neighborhood of the point $T=\gamma'$. Therefore, there exists a minimal value $\rho\in(\gamma,\gamma')$ such that $[\rho,\varphi_2(\rho)]$ is the point of inflection of the curve $y=\varphi_2(T)$. At this point, by the properties of the Jacob's ladders, we have $\varphi_2'(\rho)>0$. Let furthermore $\beta=\beta(\gamma,\rho)$ be the angle of the chord binding the points

$$[\gamma,\varphi_2(\gamma)],\ [\rho,\varphi_2(\rho)]. \tag{3.1}$$

Then we obtain by Theorem (compare [7])

Corollary 3. For every sufficiently big zero $T=\gamma$ of the function $\zeta(1/2+iT)$ the following formulae describing microscopic parts (1.7) of the Hardy-Littlewood integral (1.1) hold true

(A) a continuum of asymptotic formulae

$$\int_{\gamma}^{\gamma+U} Z^4(t)\mathrm{d}t \sim \frac{\tan\alpha}{2\pi^2} U \ln^4\gamma,\ \gamma\to\infty, \tag{3.2}$$
$$\alpha \in (0,\beta(\gamma,\rho)),\ U = U(\gamma,\alpha) \in (0,\rho-\gamma),$$

where $\alpha = \alpha(\gamma, U)$ is the angle of the rotating chord binding the points $[\gamma,\varphi_2(\gamma)],\ [\gamma+U,\varphi_2(\gamma+U)]$,

(B) a continuum of asymptotic formulae for a chord parallel to the chord given by the points (3.1)

$$\int_{N}^{M} Z^4(t)\mathrm{d}t \sim \frac{\tan[\beta(\gamma,\rho)]}{2\pi^2}(M-N)\ln^4\gamma,\ \gamma < N < M < \rho. \tag{3.3}$$

Remark 4. Let us remind that if the Riemann conjecture is true then the Littlewood estimate

$$\gamma' - \gamma < \frac{A}{\ln\ln\gamma} \to 0,\ \gamma\to\infty$$

takes place (a simple consequence of the estimate $S(T) = \mathcal{O}(\ln T/\ln\ln T)$, see [14], p. 296).

4. Second class of formulae for parts (1.7) of the Hardy-Littlewood integral (1.1) beginning in zeroes of the function $\zeta(1/2+iT)$

Let $T = \gamma, \bar{\gamma}$ be a pair of zeroes of the function $\zeta(1/2+iT)$, where $\bar{\gamma}$ obeys the following conditions (compare [7])

$$\bar{\gamma} = \gamma + \gamma^{13/14+2\epsilon} + \Delta(\gamma),\ 0 \le \Delta(\gamma) = \mathcal{O}(\gamma^{1/4+\epsilon})$$

(see the Hardy-Littlewood estimate for the distance between the neighboring zeroes [1], pp. 125, 177-184). Consequently

$$U(\gamma) = \gamma^{13/14+2\epsilon} + \Delta(\gamma) \sim^{13/14+2\epsilon},\ \gamma\to\infty. \tag{4.1}$$

For the chord that binds the points

$$[\gamma,\varphi_2(\gamma)],\ [\bar{\gamma},\varphi_2(\bar{\gamma})] \tag{4.2}$$

we obtain (similarly to [12], (5.6))

$$\tan[\alpha_2(T,U)] = 1 + \mathcal{O}\left(\frac{1}{\ln T}\right). \tag{4.3}$$

The continuous curve $y = \varphi_2(T)$ lies below the chord given by the points (4.2) on some right neighborhood of the point $T = \gamma$, and this curve lies above that chord on some left neighborhood of the point $T = \bar{\gamma}$. Therefore there exists a common point of the curve and the chord. Let $\bar{\rho} \in (\gamma, \bar{\gamma})$ be such a common point that is the closest one to the point $[\gamma,\varphi_2(\gamma)]$. Then we obtain from our Theorem (compare [7]) the next corollary.

Corollary 4. For every sufficiently big zero $T = \gamma$ of the function $\zeta(1/2+iT)$ we have the following formulae for the parts (1.7) of the Hardy-Littlewood integral (1.1)

(A) a continuum of asymptotic formulae for the rotating chord

$$\begin{aligned}(4.4)\qquad &\int_\gamma^{\gamma+U} Z^4(t)\mathrm{d}t \sim \frac{\tan\alpha}{2\pi^2}U\ln^4\gamma,\ \tan\alpha\in[\eta,1-\eta],\\ &U=U(\gamma,\alpha)\in(0,\bar{\rho}-\gamma),\end{aligned}$$

where α is the angle of the rotating chord binding the points $[\gamma,\varphi_2(\gamma)]$ and $[\gamma+U,\varphi_2(\gamma+U)]$, and $0<\eta$ is an arbitrary small number,

(B) a continuum of asymptotic formulae for the chords parallel to the chord binding the points (4.2), (see (4.3))

$$(4.5)\qquad \int_N^M Z^4(t)\mathrm{d}t \sim \frac{1}{2\pi^2}(M-N)\ln^4\gamma,\ \gamma\le N<M\le\bar{\rho}.$$

Remark 5. For example, in the case $\alpha=\pi/6$ we have from (4.4)

$$\int_\gamma^{\gamma+U} Z^4(t)\mathrm{d}t \sim \frac{1}{2\sqrt{3}\pi^2}U\ln^4\gamma,\ U=U\left(\gamma,\frac{\pi}{6}\right).$$

Remark 6. It is obvious that (see (4.4))

$$U(\gamma,\alpha)<T^{7/8+2\epsilon}.$$

Moreover, the following is also true

$$U(\gamma,\alpha)<T^{\omega+2\epsilon},\ \omega<\frac{7}{8},$$

where ω is an arbitrary improvement of the exponent 7/8 which will be proved.

Remark 7. The asymptotic formulae (1.9), (2.3), (2.4), (3.2), (3.3), (4.4), (4.5) cannot be derived within complicated methods of Balasubramanian, Heath-Brown and Ivic (compare [4]).

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References

[1] G.H. Hardy and J.E. Littlewood, 'Contribution to the theory of the Riemann zeta-function and the theory of the distribution of Primes', Acta Math. 41 (1918), 119-195.
[2] G.H. Hardy and J.E. Littlewood, 'The approximate functional equation in the theory of the zeta-function with applications to the divisor problems of Dirichlet and Piltz', Proc. Lond. Math. Soc. (2), 21, (1992), 39-74.
[3] A.E. Ingham, 'Mean-value theorems in the theory of the Riemann zeta-function', Proc. Lond. Math. Soc. 2, 27 (1926) 273-300.
[4] A. Ivic, 'The Riemann zeta-function', A Willey-Interscience Publication, New York, 1985.
[5] J. Moser, 'Jacob's ladders and the almost exact asymptotic representation of the Hardy-Littlewood integral', (2008), arXiv:0901.3973.
[6] J. Moser, 'Jacob's ladders and the tangent law for short parts of the Hardy-Littlewood integral', (2009), arXiv:0906.0659.
[7] J. Moser, 'Jacob's ladders and the multiplicative asymptotic formula for short and microscopic parts of the Hardy-Littlewood integral', (2009), arXiv:0907.0301.
[8] J. Moser, 'Jacob's ladders and the quantization of the Hardy-Littlewood integral', (2009), arXiv:0909.3928.
[9] J. Moser, 'Jacob's ladders and the first asymptotic formula for the expression of the sixth order $|\zeta(1/2+i\varphi(t)/2)|^4|\zeta(1/2+it)|^2$', (2009), arXiv:0911.1246.
[10] J. Moser, 'Jacob's ladders and the first asymptotic formula for the expression of the fifth order $Z[\varphi(t)/2+\rho_1]Z[\varphi(t)/2+\rho_2]Z[\varphi(t)/2+\rho_3]\hat{Z}^2(t)$ for the collection of disconnected sets', (2009), arXiv:0912.0130.

[11] J. Moser, 'Jacob's ladders, the iterations of Jacob's ladder $\varphi_1^k(t)$ and asymptotic formulae for the integrals of the products $Z^2[\varphi_1^n(t)]Z^2[\varphi^{n-1}(t)]\cdots Z^2[\varphi_1^0(t)]$ for arbitrary fixed $n\in\mathbb{N}$' (2010), arXiv:1001.1632.

[12] J. Moser, 'Jacob's ladders and the asymptotic formula for the integral of the eight order expression $|\zeta(1/2+i\varphi_2(t))|^4|\zeta(1/2+it)|^4$', (2010), arXiv:1001.2114.

[13] J. Moser, 'Jacob's ladders and the asymptotically approximate solutions of a nonlinear diophantine equation', (2010), arXiv: 1001.3019.

[14] E.C. Titchmarsh, 'The theory of the Riemann zeta-function', Clarendon Press, Oxford, 1951.

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